What Controls the Entrainment Rate of Dry Buoyant Thermals with Varying Initial Aspect Ratio?

HUGH MORRISON,a NADIR JEEVANJEE,b DANIEL LECOANET,c AND JOHN M. PETERSd

a National Center for Atmospheric Research, Boulder, Colorado
b Geophysical Fluid Dynamics Laboratory, Princeton, New Jersey
c Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, Illinois
d Department of Meteorology and Atmospheric Science, The Pennsylvania State University, University Park, Pennsylvania

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ABSTRACT: This study uses theory and numerical simulations to analyze the nondimensional spreading rate \( \alpha \) (change in radius with height) of buoyant thermals as they rise and entrain surrounding environmental fluid. A focus is on how \( \alpha \) varies with initial thermal aspect ratio \( A_r \), defined as height divided by width of the initial buoyancy perturbation. An analytic equation for thermal ascent rate \( w_t \), that depends on \( \alpha \) is derived from the thermal-volume-averaged momentum budget equation. The thermal top height when \( w_t \) is maximum, defining a critical height \( z_c \), is inversely proportional to \( \alpha \). The height \( z_c \) also corresponds to the thermal top height when buoyant fluid along the thermal’s vertical axis is fully replaced by entrained nonbuoyant environmental fluid rising from below the thermal. The time scale for this process is controlled by the vertical velocity of parcels rising upward through the thermal’s core. This parcel vertical velocity is approximated from Hill’s analytic spherical vortex, yielding an analytic inverse relation between \( \alpha \) and \( A_r \). Physically, this \( \alpha\text{--}A_r \) relation is connected to changes in circulation as \( A_r \) is modified. Numerical simulations of thermals with \( A_r \) varied from 0.5 to 2 give \( \alpha \) values close to the analytic theoretical relation, with a factor of \( 0.3 \) decrease in \( \alpha \) as \( A_r \) is increased from 0.5 to 2. The theory also explains why \( \alpha \) of initially spherical thermals from past laboratory and modeling studies is about 0.15. Overall, this study provides a theoretical underpinning for understanding the entrainment behavior of thermals, relevant to buoyantly driven atmospheric flows.

SIGNIFICANCE STATEMENT: Thermals, which are coherent, quasi-spherical regions of upward-moving buoyant fluid, are a key feature of many convective atmospheric flows. The purpose of this study is to characterize how thermals entrain surrounding fluid and spread out as they rise. We use theory and numerical modeling to explain why entrainment rate decreases with an increase in the initial thermal aspect ratio—the ratio of height to width. This work also explains why the nondimensional spreading rate (change in thermal radius with height) of initially spherical thermals from past laboratory and numerical modeling studies is about 0.15. Overall, this work provides a framework for conceptualizing the entrainment behavior of thermals and thus improved understanding of vertical transport in convective atmospheric flows.

KEYWORDS: Buoyancy; Entrainment; Small scale processes; Vertical motion; Vortices

1. Introduction

Thermals—coherent, isolated, quasi-spherical regions of upward-moving buoyant fluid—are a common feature of convective atmospheric flows. A key characteristic of thermals is the rate at which they increase in size as they ascend owing to entrainment of the surrounding fluid. Assuming thermal shape is self-similar (meaning that thermals do not change shape over time), dimensional analysis shows that thermal radius \( R \) is proportional to thermal top height \( z_c \), that is, \( dR/dz_c \) is constant\(^1\) (e.g., Scorer 1957). (Note that all symbols used in the paper are defined in the appendix.) Numerous laboratory and numerical modeling studies have supported this basic scaling (e.g., Scorer 1957; Richards 1961; Bond and Johari 2005; Zhao et al. 2013; Lai et al. 2015; Lecoanet and Jeevanjee 2019, hereinafter LJ2019; McKim et al. 2020; Morrison et al. 2021).

The rate of increase in \( R \) is closely related to the entrainment rate of thermals. From LJ2019, a thermal net fractional entrainment rate is defined as \( \epsilon = d(lnV)/dz_c \), where \( V \) is the thermal volume. Combined with self-similarity, this gives \( \epsilon = 3\alpha/R \), where \( \alpha = dR/dz_c \). We emphasize that \( \epsilon \) in this case is a net fractional entrainment rate because thermal volume is impacted by both entrainment (inflow of environmental fluid) and detrainment (outflow of thermal fluid). However, LJ2019 showed that detrainment is negligible for both laminar and turbulent dry, initially spherical thermals in a neutrally stable environment. Thus, \( \epsilon \) provides a close approximation for total entrainment in such conditions. An entrainment efficiency can also be defined as \( \epsilon = \epsilon R \), which gives \( \epsilon = 3\alpha \) for self-similar thermals.

\(^1\) Note that constant \( dR/dz_c \) following self-similarity and dimensional analysis is valid when there are no other physical length scales. It follows that this scaling applies to dry thermals in an unstratified environment within an infinite domain.

Corresponding author: Hugh Morrison, morrison@ucar.edu

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Thermals entrain by a process of drawing in fluid mainly from below the thermal (e.g., LJ2019; Zhao et al. 2013; McKim et al. 2020; Morrison et al. 2021). As a thermal spins up, buoyancy becomes concentrated near the center of rotation in the thermal’s toroidal circulation (i.e., vortex ring core); see Fig. 1 for a schematic of thermal structure. As a result, there is baroclinic generation of buoyancy on the outside edge of the vortex ring and destruction on the inside edge that lead to a spreading of the vortex (McKim et al. 2020). Moreover, without buoyant fluid present along the thermal’s vertical axis, circulation is nearly constant. This implies a basic constraint on the spreading rate of thermals following the principle of momentum conservation (Turner 1957). Specifically, buoyant vortex rings (which form the core of thermals) must expand over time to conserve momentum, with the rate of spread determined by the thermal-integrated buoyant forcing and the circulation. McKim et al. (2020) combined the buoyant vortex ring argument of Turner (1957) with the thermal’s vertical momentum equation to derive an analytic model for the vertical velocity of thermal top \( w_z \), \( R \), and buoyancy \( B \) at any time past spinup that does not rely on empirically determined parameters, provided \( w_z \), \( R \), and \( B \) are known at the time when the thermal is spun up.

While the basic mechanism of thermal entrainment and spreading is well understood, factors controlling the spreading rate are not. Lai et al. (2015) combined a relation between circulation, impulse (related to time-integrated buoyant forcing), and thermal spreading rate with an empirical power-law relation between normalized circulation and initial thermal aspect ratio \( A_\tau \) to predict \( \alpha \) from \( A_\tau \). They showed that variations in \( A_\tau \) for spheroidal thermals from ~0.5 to 2 lead to substantial variability in \( \alpha \), from about 0.1 to 0.3. These results are consistent with laboratory experiments reporting a similar range of \( \alpha \) (e.g., Scorer 1957; Escudier and Maxworthy 1973; Bond and Johari 2005, 2010; Zhao et al. 2013). A consensus from laboratory and numerical modeling studies is that \( \alpha \approx 0.12-0.18 \) for initially spherical thermals in an unstratiﬁed environment (e.g., LJ2019; Bond and Johari 2010; Zhao et al. 2013; Lai et al. 2015). Values are ~0.2-0.3 for initially oblate thermals with \( A_\tau < 1 \) and smaller for prolate thermals with \( A_\tau > 1 \), ~0.1-0.15 (see Fig. 17 of Lai et al. 2015). There is little sensitivity of \( \alpha \) to initial aspect ratio for \( A_\tau > 2 \) (Bond and Johari 2005). Modifying other aspects of thermal initial conditions can also produce variability in \( \alpha \), such as having an initial circulation (Escudier and Maxworthy 1973). Note that \( \alpha \) may also depend on the Reynolds number \( R_e \) of the flow, although LJ2019 showed with direct numerical simulation (DNS) that the basic mechanism of entrainment is the same for laminar and turbulent thermals (\( R_e \) of 630 and 6300, respectively), and \( \alpha \) was only ~20% higher for turbulent thermals. Their results indicate that turbulence is not necessary for entrainment and that the primary mechanism for entrainment is organized inflow controlled by the thermal’s buoyancy distribution.

The above discussion raises two important science questions. First, why do initially spherical thermals in an unstratiﬁed environment (and initially motionless) have \( \alpha \approx 0.15 \)? Why this particular value, and what are the physical mechanisms explaining it? Second, why does the spreading rate of thermals as they rise (\( \alpha \)) increase as their initial aspect (\( A_\tau \)) is decreased? To our knowledge, all previous studies have relied in some way on empirical constraint to obtain parameters, from either laboratory experiments or numerical modeling, at least during the spinup phase which is crucial for predicting \( \alpha \). In this study, we derive an expression for \( \alpha \) as a function of \( A_\tau \) that does not rely on such empirically determined parameters. The goal is to predict \( \alpha \) from \( A_\tau \) from the basic equations to provide a theoretical underpinning for understanding factors controlling the thermal spreading rate. The predicted values of \( \alpha \) are compared to those obtained from numerical simulations of thermals over a range of \( A_\tau \).

In the theoretical part, we first derive an analytic expression for thermal ascent rate \( w_z \) from the nondimensional thermal momentum budget equation. We then use this expression to derive an analytic relation between \( \alpha \) and the thermal spinup height \( z_c \) (defined as the thermal top height when \( w_z \) reaches a maximum), valid over a range of initial thermal aspect ratios. We show that \( z_c \) also corresponds to the time for parcels initially near the thermal bottom to ascend through the thermal core to near the thermal top. This determines the time for buoyancy to be removed from the central thermal core by entrainment of nonbuoyant environmental fluid, after which circulation is nearly constant. This time scale for thermal spinup depends linearly on \( A_\tau \) and is determined by the thermal’s internal flow structure which is well modeled by Hill’s analytic

![Fig. 1. Schematic diagram of a vertical cross section through the thermal center. The central vertical axis is indicated by the dashed line. Red X symbols mark the center of circulation comprising the vortex ring core with radius \( R_v \). After spinup, the region of nonzero buoyancy indicated by blue shading is confined to the vortex ring core. Baroclinic generation and destruction of vorticity associated with this buoyancy structure leads to outward spreading of the vortex ring structure and thermal as a whole as shown by the red arrows. Black curved lines illustrate streamfunction isolines (only shown for the right half of the thermal). The thermal boundary, which is also a streamfunction isoline, is indicated by the curved blue line. This boundary also defines the thermal radius \( R \).](image-url)
spherical vortex (Hill 1894) even for nonspherical thermals. We show that the predicted values of \( w_r, \alpha, \) and \( z \) are consistent with numerical simulations of buoyant thermals over a range of \( A_t \) from 0.5 to 2.

The paper is organized as follows. Section 2 provides a theoretical description of the problem and derivation of equations for \( w_r \) and \( \alpha \). Section 3 gives a description of the numerical model and experimental design. Section 4 presents results from the numerical simulations and comparison of these results with theory. A summary and conclusions are given in section 5.

2. Theoretical description

We first write the basic governing equations that set the stage for the rest of the derivation. These are the incompressible Boussinesq–Euler equations for fluid motion and mass continuity plus the conservation equation for perturbation fluid density:

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\rho_0^{-1} \nabla p - \rho' \frac{\rho_0}{\rho},
\]

\[
\nabla \cdot \mathbf{u} = 0,
\]

\[
\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho' = 0,
\]

where \( t \) is the time, \( \mathbf{u} \) is the velocity vector, \( \mathbf{k} \) is a unit vector pointed in the vertical (opposite to the direction of gravitational acceleration), \( p \) is the pressure, \( \rho_0 \) is a constant background fluid density, \( \rho' \) is a perturbation fluid density from the background density, and \( g \) is the gravitational acceleration. Here, the environment is assumed to be neutrally stable; for inviscid flow in a neutrally stable environment and applying the incompressible Boussinesq approximation, the perturbation density acts as a fluid tracer with no sources or sinks. In the following, we use buoyancy defined as \( B = -g \rho'/\rho_0 \).

a. Thermal momentum budget

To derive our analytic thermal model and theoretical values for \( \alpha \), we first focus on the vertical momentum budget of thermals. This allows us to derive an analytic expression for the thermal ascent rate, from which we obtain expressions for thermal spreading rate \( \alpha \) later. Defining the thermal as occupying some portion of space defined by \( \Omega \) within the domain, we can integrate over \( \Omega \) to obtain the thermal’s budget of vertical momentum \( \rho_0 w \), where \( w \) is the vertical velocity. Because \( \Omega \) changes over time, we use Gauss’s theorem to relate the divergence of the momentum field over \( \Omega \) to the flux of momentum across the surface of \( \Omega \) (Romps and Charn 2015; Morrison et al. 2022):

\[
\frac{d}{dt} \int_{\Omega(t)} \rho_0 w d^3x = -\int_{\partial\Omega(t)} \frac{\partial \rho}{\partial z} d^3x + \int_{\Omega(t)} \rho_0 B d^3x
\]

\[
+ \int_{\partial\Omega(t)} (\mathbf{u} \cdot \mathbf{n}) \rho_0 w d^2x,
\]

where \( \partial \Omega(t) \) is the two-dimensional boundary of \( \Omega(t) \), \( \mathbf{n} \) is a unit vector normal to the thermal’s surface, and \( \mathbf{u} \) is an effective entrainment velocity defined as the displacement rate of the thermal boundary \( \mathbf{u} \) relative to flow velocity \( \mathbf{u} \), i.e., \( \mathbf{u} = \mathbf{u}_w - \mathbf{u} \).

Defining \( \mathbf{v} = V^{-1} \int_{\Omega(t)} w d^3x \) as the thermal-averaged \( w \) (assumed to be equal to the thermal top ascent rate \( w_t \)) in (4), using the product rule \( \int (w V/\partial t) = \int V dwdx + \int w V dV/dt \), dividing by thermal volume \( V \), and rearranging terms, we express the thermal-averaged momentum budget as

\[
\rho_0 \frac{d w_t}{dt} = -F_d + E + \rho_0 \mathcal{B},
\]

where \( F_d = V^{-1} \int_{\Omega(t)} (\rho p/\partial z) d^3x \) is the thermal-averaged pressure drag force, \( E = V^{-1} \int_{\Omega(t)} (\mathbf{u} \cdot \mathbf{n}) \rho_0 w^2 dx - V^{-1} \int_{\Omega(t)} (dV/dt) w \rho_0 \) is the momentum entrainment, and \( \mathcal{B} = V^{-1} \int_{\Omega(t)} B d^3x \) is the thermal-averaged buoyancy. Equation (5) expresses the vertical momentum budget of a thermal as an acceleration term on the left-hand side and a drag force arising from vertical pressure gradients, an entrainment “pseudoforce,” and a buoyant forcing term on the right-hand side.

We assume that detrainment is negligible and that thermal expansion incorporates fluid with \( w = 0 \). Thus, net entrainment is related only to the change in thermal volume (following the Boussinesq approximation). This assumption is well justified based on the DNS of dry thermals from LJ2019; see Morrison et al. (2022) for further discussion. It follows that we can write the entrainment term as \( E = V^{-1} (dV/dt) w \rho_0 \), where \( V^{-1} (dV/dt) w \) is an effective vertical velocity. Using the chain rule. Thus, we can express \( E = -\omega^2_0 \rho_0 \).

We nondimensionalize the thermal momentum equation next. Based on a characteristic background fluid density \( \rho_0 \), thermal radius \( R_0 \), and thermal buoyancy \( B_0 \), we can define various scales including time \( t_0 = \sqrt{R_0/B_0}\), velocity \( w_0 = \sqrt{R_0/B_0} \), and pressure \( p_0 = \rho_0 B_0 R_0 \). From this, we write \( t \equiv t_0 \omega, \ z \equiv z/R_0, \ w \equiv w/w_0, \ F_d \equiv (R_0/p_0) F_d, \ E \equiv E/E_0 = [R_0/(\rho_0 w_0^2)]E_0, \) and \( B^* = B/B_0 \). Substituting these relations into the thermal-averaged momentum budget gives a nondimensional form of the equation:

\[
\frac{d w_t}{dt} = \frac{p_0 w_0^2}{R_0} \frac{d w_t^*}{dt} = \frac{p_0 w_0^2}{R_0} F_d^* + \frac{p_0 w_0^2}{R_0} E^* + \rho_0 B_0 \mathcal{B}^*.
\]

Multiplying (6) by \( R_0/(w_0^2 \rho_0) \), it can be expressed in terms of two nondimensional flow parameters: Froude number \( F_r = w_0^2/(B_0 R_0) \) and Euler number \( E_r = \rho_0/(\rho_0 w_0^2) \). This gives

\[
\frac{d w_t^*}{dt} = -E_r F_{pB}^* - E_r F_{dB}^* + E^* + F_r \mathcal{B}^*,
\]

where the nondimensional drag force \( F_{dB}^* \) is divided into two parts following the standard separation of perturbation pressure into buoyant and dynamic components: \( F_{dB}^* = F_{pB}^* + F_{dB}^* \). The buoyant part is approximated by \( E_r F_{pB}^* \approx F_r^{-1}(1 - C_v \mathcal{B}) \). The term \( C_v \) is a virtual mass parameter defined such that the sum of \( F_r^{-1} \mathcal{B} \) and \( -E_r F_{pB}^* \) is equal to \( F_r^{-1} C_v \mathcal{B} \). \( C_v \) depends on the structure of the buoyancy field. For example, \( C_v = 2/3 \) for a spherical buoyancy perturbation (Tarshish et al. 2018). It follows that the nondimensional effective buoyancy—the sum of buoyancy and buoyant perturbation pressure forcing—is \( B_{eff}^* = C_v \mathcal{B} \).
We approximate the dynamic pressure part of thermal drag using the standard drag equation divided by $V$ to give a volume-averaged dynamic pressure drag force: $F_{dp} = \rho w^2 C_d A/(2V)$, where $A$ is the cross-sectional thermal area perpendicular to the flow. Defining $\gamma$ as the ratio $AR/V$, which is a constant for self-similar thermals ($\gamma = 3/4$ for a spherical thermal), we can express the thermal-averaged nondimensional dynamic pressure drag force as $F_{dp*} = w^2 \gamma C_d/(2R^2)$. Note that $C_d \approx 0$ for initially spherical ($A_t = 1$) dry buoyant thermals (Morrison et al. 2022). Although $C_d$ could in principle vary with $A_t$, the thermal simulations presented in section 4, with $A_t$ varying from 0.5 to 2, all have small $C_d$ (magnitude less than 0.1). Thus, the dynamic pressure drag is relatively unimportant in the thermal momentum budget. Nonetheless, we retain this term and $C_d$ in the equations for generality.

Hereinafter, we take characteristic values of the physical scales $\rho_0$, $R_0$, and $B_0$ as unity so that $F_r = 1$ and $E_u = 1$ and drop the * indicating nondimensional quantities for convenience. It follows that we can write the nondimensional thermal-averaged vertical momentum budget equation given by (7) as

$$\frac{1}{2} \frac{d w^2}{d z_t} + w^2 \gamma + \frac{C_d w^2}{2R} - C_d B = 0,$$

where we have used the above relations for $E$ and buoyant and dynamic pressure contributions to drag, and the chain rule to express the time derivative as a height derivative following the thermal top: $\frac{d}{dt} = w \frac{d}{dz_t}$ ($z_t$ is the thermal top height).

If we assume $\alpha = dR/dz_t$ is constant, consistent with recent numerical modeling studies (Morrison et al. 2021; LJ2019) and the simulations herein, we can write $R = R_0 + \alpha z_t = 1 + \alpha z_t$ by integrating $\alpha$ from $z_t' = 0$ to $z_t = z_t$. Also, $B = B_0/R^3 = 1/R^3$ (since we take $B_0 = 1$ as the initial buoyancy scale, and $B$ scales inversely with the change in thermal volume as the thermal entrains non-buoyant environmental fluid and expands) and $\epsilon = \epsilon/R$ (LJ2019). With these assumptions and relations, (8) may be written as

$$\frac{1}{2} \frac{dw^2}{d z_t} + \left(\frac{\gamma C_d}{1 + \alpha z_t} \right) \frac{w^2}{2R} - C_d B = 0.$$

Equation (11) is similar to Escudier and Maxworthy [1973, Eqs. (9)–(11) therein], except that we invoke the Boussinesq approximation, express $w_t$ using a single analytic equation as a function of height $z_t$ rather than $t$, and include an explicit dependence on $\alpha$.

Solutions for $w_t$ can be obtained from (11), provided values of $\alpha$, $\epsilon$, $C_d$, and $C_v$ are known. Past literature has suggested $\alpha \approx 0.5–0.8$ (Lai et al. 2015), $\epsilon \approx 3\alpha$ (LJ2019), $C_d \approx 0$ (Morrison et al. 2022), meaning that $\gamma$ is not relevant, and $C_v \approx 0.5–0.8$ (Tarshish et al. 2018). Examples of solutions to the analytic $w_t$, Eq. (11) using $\epsilon = \alpha$, $C_d = 0$, $C_v = 2/3$, and $\alpha$ ranging from 0.05 to 0.3 are shown in Fig. 2 (solid lines). With these parameter values, (11) gives a family of solutions that all exhibit a sharp increase of $w_t$ with height initially corresponding to a “slippery” regime when upward buoyant forcing is primarily balanced by vertical acceleration, followed by a slower decrease of $w_t$ corresponding to a “sticky” regime when weak buoyant forcing is balanced mainly by entrainment. This shape of the $w_t$ profile with the two distinct regimes of thermal evolution was discussed previously via analysis of numerical solutions (e.g., Wang 1971; Tarshish et al. 2018). In subsequent sections, we will determine constraints on values of $\alpha$ while also briefly exploring how $\epsilon$ and $C_v$ vary with the initial thermal aspect ratio.

We can understand scaling behaviors in the slippery and sticky regimes via asymptotic analysis and expansion of (11), similar to the asymptotic analysis of Escudier and Maxworthy (1973) applied to their equation set for $w_t$. For the slippery regime when $z_t < 1$, we can expand (11) using Taylor series

2 Note that the sticky regime is primarily a balance between entrainment and buoyant forcing for dry thermals, whereas Romps and Charn (2015) identified a sticky regime for cloud thermals consisting mainly of a balance primarily between buoyant forcing and downward pressure gradient forcing.
about \( z_t = 0 \) and retain the first-order term to give \( w_t \approx z_t^{-1/2} \).
This is equivalent to a scaling of \( z_t \propto r^2 \) since \( w_t = dz_t/dt \), which is consistent with the theory and numerical simulations in the slippery regime from Tarshish et al. (2018).

In contrast, when \( z_t \gg 1 \) for the sticky regime, \( 1 + \alpha z_t \approx \alpha z_t \) and the second term on the right-hand side of (11) is negligible compared to the first term (for \( 2\epsilon/\alpha + \gamma C_f/2 > 2 \), which is satisfied for typical values of \( 2\epsilon/\alpha \approx 6 \) and \( \gamma C_f \approx 1 \), implying a scaling of \( w_t \) with \( z_t^{-1} \). This is equivalent to a scaling of \( z_t \propto r^2 \), consistent with the theory and simulations of LJ2019 for the sticky regime. These scaling regimes well correspond to full solutions of the analytic \( w_t \) equation, Eq. (11), for small and large \( z_t \) as seen in Fig. 2 and are also consistent with the asymptotic analysis from Escudier and Maxworthy (1973).

b. Impulse, circulation, and thermal spreading rate

Although (11) accounts for the impact of thermal spreading rate \( \alpha \) on \( w_t \) via entrainment, by itself it does constrain \( \alpha \). However, the thermal momentum budget can provide a basic constraint on \( \alpha \) via the impulse–circulation relation (e.g., Turner 1957; Lai et al. 2015; McKim et al. 2020; Morrison et al. 2021). A thermal’s circulation \( \Gamma \) can be calculated as the integral of azimuthal vorticity \( \omega_0 \) over \( S \) in axisymmetric \((r, z)\) coordinates (McKim et al. 2020):

\[
\Gamma = \oint_S u \cdot dl = \int_S \omega_0 \, dr \, dz. 
\]

The time rate of change of circulation is given by

\[
\frac{d\Gamma}{dt} = \int_S \frac{\partial B_c}{\partial r} \, dr \, dz = -\int_{z_0}^{z_e} B_c \, dz \tag{13}
\]

for an inviscid fluid, where \( B_c \) is the buoyancy along the thermal’s vertical axis, \( z_0 \) and \( z_e \) are the bottom and top heights of the region \( S \), and any region with \( B_c > 0 \) is assumed to be encompassed within \( S \) (hereinafter when writing this integral, we drop the limits \( z_0 \) and \( z_e \) for convenience). In words, (13) shows that the rate of increase in \( \Gamma \) during spinup is equal to the vertical integral of core buoyancy.

We can directly relate \( \Gamma \) to \( \alpha \) via the impulse–circulation relation. Fluid impulse \( I \) is the total momentum change starting from rest caused by external forcing over a finite region of space. For an idealized infinite domain in the absence of nonconservative forces and applying the incompressible Boussinesq approximation, \( I \) is related to vorticity \( \omega \) by (e.g., Batchelor 2000)

\[
I = \frac{\rho_0}{2} \oint_V x \times \omega \, dl, \tag{14}
\]

where \( V \) is the entire domain and \( x \) is a position vector. The term \( I \) is essentially a record of the volume- and time-integrated external (here, gravitational via a localized buoyancy anomaly) forcing on the fluid. If the region of nonzero vorticity within the domain is concentrated on a circle with a radius equal to \( R_c \) (i.e., the vortex ring radius, see Fig. 1), and this region of vorticity is small compared to \( R_c \), then (14) can be approximated using the second part of (12) as (Turner 1957; McKim et al. 2020)

\[
I = \pi \rho_0 \Gamma R_c^2. \tag{15}
\]

The subscript “z” indicates that the impulse is only in the \( z \) direction as a consequence of the axisymmetry of the flow.

Taking \( d\Gamma/dt \) of (15), we have

\[
\frac{d\Gamma}{dt} = \pi \rho_0 \left( R_c^2 \frac{dR_c}{dt} + \Gamma \frac{dR_c^2}{dt} \right) = F_B. \tag{16}
\]

where \( F_B \) is the domain-integrated buoyant forcing, which is constant in time following (3) in a neutrally stable environment.

Equation (16) can be directly related to \( \alpha \) using the chain rule \( d\Gamma/dt = w_d dz_t \) to give \( dR_c^2/dt = \xi^2 d\xi^2/dt = 2\xi^2 R_c \alpha \), where \( \xi \) is the ratio of the vortex radius to the thermal radius and assumed to be constant following self-similarity but may vary with \( A_r \). Substituting this relation for \( dR_c^2/dt \) in (16) and substituting (13) for \( d\Gamma/dt \), we have

\[
\pi \rho_0 \xi^2 R_c^2 \int B_c \, dz + 2 \pi \rho_0 \Gamma \xi^2 R_c \alpha = F_B. \tag{17}
\]

The physical interpretation of (13)–(17) is that if there is no buoyancy along the thermal’s vertical axis (\( B_c = 0 \), then there is no change in the thermal’s circulation with time \( (d\Gamma/dt = 0) \) and the first term on the left-hand side of (17) is zero. The removal of buoyancy along the thermal’s vertical axis by the upward encroachment of nonbuoyant environmental fluid entrained from below the thermal corresponds to the point of thermal spinup. With \( F_B > 0 \), \( B_c = 0 \) implies \( \alpha > 0 \) in the second term since \( \xi, R \), and \( w_t \) are all \( > 0 \). This is consistent with the basic argument from Turner (1957) and McKim et al. (2020) explaining how the impulse–circulation relation dictates a positive thermal expansion rate when \( d\Gamma/dt = 0 \) and \( F_B > 0 \). Moreover, \( F_B \) is constant as noted above, and self-similarity implies \( \alpha \) and \( \xi \) are constants and \( R \propto z_t \). This gives \( w_t \propto z_t^{-1} \), which is consistent with the scaling from our analytic \( w_t \) equation, Eq. (11), in the “sticky” regime where \( z_t \gg 1 \). During the spinup phase corresponding to the “slippery” regime, \( B_c > 0 \) and the first term on the left-hand side of (17) > 0. This implies that the second term on the left-hand side of (17) must be smaller before spinup than after.

If \( \Gamma \) is known when the thermal is spun up, we can derive an analytic expression for \( \alpha \) by combining (15) and (17) with the relation between a thermal’s impulse and \( w_t \) (Akhetov 2009; McKim et al. 2020): \( I = mR^3 \rho_0 (1 + C_w) w_t \), where \( m \) is a shape parameter equal to the ratio of thermal volume to \( R^3 \).

This is analogous to Eq. (17) in McKim et al. (2020), who expressed \( e \) (rather than \( \alpha \)) in terms of \( \xi, F_B, \Gamma, C_w \), and \( m \). All of these parameters likely depend on \( A_r \). For example, the virtual mass coefficient \( C_w \) depends on the thermal shape (Tarshish et al. 2018), while \( \Gamma \) (after spinup) has a strongly nonlinear dependence on \( A_r \) as we show in section 4 from the thermal numerical simulations. Such dependencies are broadly consistent.
with sensitivity of \( \alpha \) to \( A_r \), but complicate the interpretation and explanation of this sensitivity.

These complications motivate an alternative approach described below that relates \( \alpha \) to the thermal top height when spinup is achieved, corresponding to the thermal top height when \( w_c \) reaches its maximum. This is in a similar vein as relating \( \alpha \) (or \( e \)) to \( \zeta \), \( F_b \), \( \Gamma \), \( C_w \), and \( m \), but with much simpler functional dependencies allowing for a clear understanding of the variation of \( \alpha \) with \( A_r \).

c. Relationship between thermal spinup height and spreading rate \( \alpha \)

We define a critical thermal top height \( z_c \) separating the “slippery” and “sticky” regimes when \( dw_c/dz_t = 0 \) and \( w_c \) is a maximum. During spinup, \( dw_c/dz_t > 0 \) owing to the presence of buoyant fluid along the thermal’s vertical axis. After buoyancy is eroded along the thermal’s vertical axis from entrainment of environmental fluid, \( \Gamma \) is constant following (13), and thus, \( dw_c/dz_t < 0 \) following the \( w_c \propto z_c^{-1} \) scaling implied by (17). Taken together, the implication is that \( z_c \) corresponds to the thermal top height at the time when core buoyancy is eroded and thereafter \( d\Gamma/dt \sim 0 \), which is supported by the numerical simulations presented in section 4.

The critical height \( z_c \) is obtained by taking \( d\Gamma/dz_t \) of (11) and setting \( dw_c/dz_t = 0 \). We also introduce the parameter \( b = e/\alpha \), the ratio of entrainment efficiency to thermal spreading rate. The resulting expression is solved analytically to yield the following relation between \( z_c \), \( \alpha \), \( b \), \( \gamma \), and \( C_d \):

\[
(b + \frac{\gamma C_d}{2\alpha})(1 + \alpha z_c)^{-2b - (\gamma C_d/\alpha)} = 1.
\]

Figure 3 shows \( \alpha \) calculated from (18) as a function of \( z_c \) for \( b \) equal to 2, 2.5, and 3 and \( C_d = 0 \) (Fig. 3a) and for \( C_d \) equal to \(-0.1 \), 0, and 0.1 and \( b = 3 \) (Fig. 3b). For these calculations, \( \gamma = 3/4 \) corresponding to spherical thermals for simplicity. A self-similar thermal shape implies \( b = 3 \), meaning that \( e = 3\alpha \) and \( e = d(\ln \Gamma)/dz_c = 3\alpha/R \) (LJ2019). Deviations from \( b = 3 \), therefore, indicate the degree to which thermals are not self-similar. Zhao et al. (2013), while noting self-similarity of gross thermal characteristics (overall thermal shape and size) after spinup, found that internal vorticity and density structures evolved nonsimilarly in their experimental study. In the simulations presented later in the current paper, \( b \) is somewhat smaller than 3, ranging from \(-2.4 \) to \( 2.7 \) for \( A_r < 2 \). For the \( A_r = 2 \) simulation, \( b \sim 2.1 \), indicating that initially vertically elongated thermals change their shape relatively more than the smaller \( A_r \) thermals. This may be related to the inability of thermals to take in all initially buoyant fluid when \( A_r < 2 \), leaving a wake of buoyant fluid below the thermal, as discussed in Lai et al. (2015). Nonetheless, sensitivity of \( \alpha \) to \( b \) over the range of 2–3 is fairly small, with a 24% decrease in \( \alpha \) as \( b \) is increased from 2 to 3.

The change in \( \alpha \) for a given \( z_c \) as \( C_d \) is varied from \(-0.1 \) to 0.1 is small in magnitude, with \( \alpha \) varying by up to 0.04 for the range of parameters shown in Fig. 3b. The relative change is greatest at small values of \( \alpha \) (large \( z_c \)), up to \(-50\% \) in Fig. 3b. For the thermal numerical simulations detailed later in the paper, with \( A_r \) varying from 0.5 to 2, mean \( C_d \) ranges from \(-0.06 \) to \( 0.08 \) [using the same method to calculate \( C_d \) from the simulated dynamic perturbation pressure field as in Morrison et al. (2022)]. Thus, dynamic pressure drag is relatively unimportant in (18), and hereinafter, we will assume \( C_d = 0 \). With this assumption, (18) can be rearranged to give

\[
\alpha = \frac{e^{(3b-2)} - 1}{z_c}.
\]
If we further assume a self-similar thermal shape, we can use (19) with \( b = 3 \) to obtain
\[
\alpha = \frac{3^{1/4} - 1}{z_c}. \tag{20}
\]

The one-to-one relationship between \( \alpha \) and \( z_c \) means that if \( z_c \) is known, this uniquely constrains the value of \( \alpha \). We emphasize that \( z_c \) does not cause a particular value of \( \alpha \), but if \( z_c \) is known, then \( \alpha \) can be predicted from it. A key point is that the expressions for \( \alpha \) in (18)–(20) are independent of \( C_p \) and thus expected to have little dependence on \( A_r \) (given that \( b \) does not vary much with \( A_r \) and \( C_d \approx 0 \)). Equation (20) gives consistent results with the analytic thermal profiles shown in Fig. 2, which have \( z_c \) ranging from \(-1\) to \(6 \) for \( \alpha \) of \(0.05 \)–\(0.3\) (for \( b = 3 \) and \( C_d \approx 0 \)).

d. Relationship between \( z_c \) and \( A_r \)

As argued in the previous subsection, \( z_c \) corresponds to the thermal top height when buoyant fluid along the thermal’s vertical axis is replaced by entrained environmental fluid (meaning circulation is approximately constant thereafter). This erosion of buoyancy in the thermal core occurs as nonbuoyant parcels are entrained near the thermal bottom and move upward relative to the thermal as a whole. Thus, we expect the time scale for loss of buoyancy along the thermal’s vertical axis to be equal to the time for parcels entrained near the thermal bottom to travel upward through the thermal.

A parcel must travel a distance of the initial thermal depth plus \( z_c \) to ascend through the thermal in the same amount of time as the thermal top takes to reach height \( z_c \). Since we can express the initial thermal depth as \( D_0 = 2A_r \) (keeping in mind \( R_0 = 1 \)), this time scale is
\[
\tau_c = z_c + 2A_r, \tag{21}
\]
where \( \tau_c \) is the time-averaged parcel vertical velocity along its Lagrangian path: \( \tau_c = \frac{1}{\tau_c} \int_0^{z_c} w_p(t)dt \). By definition, this is the same time scale for the thermal top to reach \( z_c \) (starting from \( z_c = 0 \)), implying
\[
\tau_c = z_c, \tag{22}
\]
where \( \tau_c = \frac{1}{\tau_c} \int_0^{z_c} w(t)dt \) is the time-averaged thermal top vertical velocity.

Substituting (21) in (22) and solving for \( z_c \) gives
\[
z_c = \frac{2A_r}{\sigma - 1}. \tag{23}
\]
where \( \sigma = \tau_c/\tau_p \) is the ratio of the time-averaged vertical velocities of the parcel and thermal top.

Equation (23) can be combined with (19) to yield an expression for \( \alpha \) as a function of \( A_r \);
\[
\alpha = \frac{(3^{1/4} - 1)(\sigma - 1)}{2A_r}. \tag{24}
\]

If we take \( b = 3 \) following self-similarity, this gives
\[
\alpha = \frac{(3^{1/4} - 1)(\sigma - 1)}{2A_r}. \tag{25}
\]

e. Predicting \( \sigma \) from Hill’s analytic spherical vortex

The thermal aspect ratio \( A_r \) is specified from the initial conditions, leaving \( \sigma \) as the only unknown parameter in (25) to obtain \( \alpha \). This parameter is closely related to the thermal internal flow structure, which controls the rate of parcel ascent in the thermal core relative to the thermal as a whole. Lai et al. (2015) noted similarity of the flow structure of thermals to Hill’s vortex, particular for \( A_r \approx 2 \). They found that the analytic Hill’s vortex solution deviated more from numerical thermal simulations for smaller \( A_r \), but noted “it can still give a fair prediction of flow field” for \( A_r \) as low as \(0.5 \). In agreement with Lai et al. (2015), in section 4, we show a close correspondence of vertical velocity profiles along the central axis in numerically simulated thermals to Hill’s vortex for initial \( A_r \) of \( 1 \) and \( 2 \), with more deviation but still fairly similar \( w \) profiles for \( A_r \approx 0.5 \).

Given the overall similarity of Hill’s analytic vortex with the internal flow of thermals, we can approximate \( \sigma \) from the vertical profile of \( w \) in the core of Hill’s vortex. The \( w \) field within Hill’s vortex is given by
\[
w(r, z) = \frac{3W}{4} \left( \frac{r}{a} \right)^2 + \frac{r^2}{a} - \frac{10}{3}, \quad \forall z^2 + r^2 \leq a, \tag{26}
\]
in axisymmetric coordinates, where \( a \) is the vortex radius and \( W \) is the steady vortex ascent rate. The flow outside of the vortex is given by
\[
w(r, z) = \frac{W}{2} \left( \frac{r^2}{z^2 + r^2} \right)^{5/2}, \quad \forall z^2 + r^2 > a. \tag{27}
\]
Along the vertical axis (\( r = 0 \)), the \( w \) profile is symmetric and features an increase in the bottom half of the vortex, a maximum \( w \) equal to \( 5/2W \) at \( z = 0 \), and a decrease in the upper half.

A parcel initially at the bottom of Hill’s vortex will rise at the same rate as the vortex since \( u = 0 \) and \( w = W \) at this location (i.e., it is a stagnation point in the vortex-relative flow). However, a parcel initially just above the vortex bottom at \( r = 0 \) will rise relative to the vortex as a whole. Thermals, owing to their buoyancy, entrainment, and nonsteady behavior, do not have such stagnation points, and parcels initiated at the thermal bottom rise through the thermal depth as demonstrated by the simulations in section 4. Thermal flow is similar to Hill’s vortex in the interior. Thus, although parcel ascent differs between thermals and Hill’s vortex near the top and bottom boundaries, it is similar in the interior with an acceleration toward the center followed by a deceleration above.

Because of the stagnation points in Hill’s vortex, we cannot use it directly to estimate the Lagrangian time scale for parcel ascent starting from the thermal bottom. However, given similarity of the interior flow between thermals and Hill’s vortex,
a rough approximation is to replace the Lagrangian time-mean \( w \) along the parcel’s path with the Eulerian vertical-mean \( w \) from Hill’s vortex: \( \bar{w}_p = (2a)^{-1} \int_{z=0}^{z=a} w(z)dz = 2W \), where \( w(z) \) is from (26) with \( r = 0 \). This gives \( \sigma = \bar{w}_p/W \approx 2 \), which is expected to be an upper estimate since the Lagrangian mean weights toward smaller values of \( w \) compared to the Eulerian mean. Additional context for this approach is provided by analysis of the thermal numerical simulations. Comparing the Lagrangian mean \( \bar{w}_p \) for a parcel initiated at the thermal bottom versus the time-averaged Eulerian mean \( w \) (from thermal bottom to top) during the spinup period shows a close correspondence between the two, with relative differences ranging from −6% to 14%. Furthermore, \( \sigma \) values from the simulations generally range from 1.80 to 1.95 (with the exception of \( \sigma \approx 1.63 \) in the \( A_\nu = 2 \) simulation), close to but slightly less than \( \sigma = 2 \).

We can also calculate \( \bar{w}_p \) from Hill’s vortex exactly for a parcel initiated above the vortex bottom and ending the same distance below vortex top. This is obtained from

\[
\bar{w}_p = (D + 2fa)/\Delta t, \tag{28}
\]

where \( f \) is the fractional distance from the vortex center (\( z = 0 \)) where the parcel is initiated relative to its radius \( a \), \( \Delta t \) is the time for the parcel to travel along this path, and \( D = W\Delta t \) is the distance traveled by the vortex as a whole over \( \Delta t \). The distance \( D + 2fa \) is the total distance traveled by the parcel over its Lagrangian path. Following a trajectory along \( r = 0 \), \( dz/dt = w(z) - W \), where \( z \) is height relative to the ascending vortex and \( w(z) - W \) is the vortex-relative parcel velocity. The time scale for ascent is calculated as \( \int_{z=0}^{z=\Delta t} dt = \Delta t = \int_{z=0}^{z=a} [w(z) - W]^{-1} dz \). The integral on the right-hand side can be solved analytically by substituting (26) for \( w(z) \) combined with \( r = 0 \) to yield

\[
\Delta t = -2a/3W[ln(1 - f) - ln(1 + f)]. \tag{29}
\]

Combining (28) and (29) with \( D = W\Delta t \) gives an expression for \( \bar{w}_p \), and \( \sigma \) is then obtained by dividing this expression by \( W \) to give

\[
\sigma = 1 - 3f[ln(1 - f) - ln(1 + f)]^{-1}. \tag{30}
\]

A parcel initiated just above the vortex bottom, with \( f \) of 0.99–0.9 (i.e., initiated at a distance of 0.01–0.1 radii above the thermal bottom and ending the same distance below top), gives \( \sigma \) of 1.6–1.9 consistent with the simulations.

Following discussion in Lai et al. (2015), the flow field of the Norbury vortex family (Norbury 1973), which generalizes Hill’s vortex to variable ring vortex thickness, may be closer to the thermal simulations with varying \( A_\nu \). Similarly, the O’Brien (1961) analytic spheroidal vortex model might give a better description of the flow for spheroidal thermals. However, these models are steady state and also have stagnation points. Since Hill’s vortex provides a reasonable description of the interior thermal flow over a range of \( A_\nu \), we use it to constrain \( \sigma \) following the discussion above.

Combining \( \sigma \approx 2 \) from the Eulerian mean \( w \) of Hill’s vortex with (19) and (23) gives our final theoretical expression for \( \alpha \) (with the assumption of self-similarity so that \( b = 3 \)):

\[
\alpha \approx (3^{1/3} - 1) 0.158 \approx A_\nu. \tag{31}
\]

3. Description of the numerical simulations

a. Model description and experimental design

We utilize the Cloud Model 1 (CM1) fluid flow model to numerically simulate thermals with varying initial \( A_\nu \). CM1 is a nonhydrostatic model which has been widely used to simulate idealized atmospheric flows. Here, we use the incompressible Boussinesq configuration to solve the filtered Navier-Stokes equations similar to the large-eddy simulation (LES) configuration in Morrison et al. (2022). Prognostic variables are the 3D components of flow velocity and potential temperature perturbation \( \theta' \), although near-axisymmetry of the model fields is retained. Buoyancy \( B \) is obtained by \( \rho'\theta_0 \), where \( \theta_0 \) is a constant background \( \theta \) of the fluid environment. As noted by Morrison et al. (2022), in this framework, prognosing \( \theta' \) is equivalent to prognosing \( \rho' \) itself. Simulations are nondimensionalized using a length scale equal to the radius of the initial thermal \( R_0 \) (the radius of the initial buoyancy perturbation) and a time scale given by \( \sqrt{R_0/B_0} \), where \( B_0 \) is the initial thermal buoyancy. The density scale \( \rho_0 \) is equal to the constant background fluid density in this Boussinesq framework. All other quantities are nondimensionalized following these basic scales.

The initial \( A_\nu \) of thermals is varied from 0.5 to 2, similar to the range from Lai et al. (2015). As we show in section 4, this produces a wide spread of \( \alpha \) (~0.08–0.25). Thermals are initiated by adding a buoyancy perturbation \( B_0 \) uniformly within a spheroidal volume having a horizontal radius of \( R_0 \) and a vertical radius of \( A_\nu R_0 \). To minimize the impacts of boundary conditions, the initial buoyancy perturbations are centered at a height of \( 4R_0 \) and the horizontal domain width is \( \approx 16R_0 \). The vertical domain height is \( 64A_\nu R_0 \) (64 times the initial vertical thermal radius). The model grid is isotropic in all three directions with a grid spacing \( \Delta L_m \) equal to \( 0.1A_\nu R_0 \). Since the initial \( A_\nu \) varies from 0.5 to 2, \( \Delta L_m \) ranges from 0.05 to 0.2 \( R_0 \). The time step is 0.0362 times the time scale \( \sqrt{R_0/B_0} \). Because the thermals expand as they ascend, the overall dynamical structure is well resolved with at least 10 grid points horizontally and 20 points vertically across the thermals. An additional set of simulations with \( A_\nu \) varying from 0.5 to 2 but \( \Delta L_m = 0.1R_0 \) (thus at least 20 points horizontally and 10 points vertically across the thermals) was also run and analyzed. This set gives similar results to the first set, and thus, we only report the results of the first set of simulations in this paper. Other details of the model setup are given in Table 1.

In this study, we use LES applied to the filtered Navier–Stokes equations instead of DNS to retain a close connection to atmospheric modeling, particularly modeling of dry and moist thermals in the planetary boundary layer and
convective clouds in which DNS is not possible given the huge $O(10^9)$ Reynolds numbers involved. Our LES framework is also consistent with our previous work on dynamic drag of dry buoyant thermals (Morrison et al. 2022) and similar to previous thermal simulations of Lai et al. (2015). The subgrid-scale (SGS) mixing follows a Smagorinsky-type approach as implemented by Stevens et al. (1999, see their appendix B, section b). The SGS mixing length is set to $\Delta L_m$. Because the dissipation scale (the model’s filter scale) is a relatively large fraction of the thermal’s radii, the resolved scale flow is smooth and thus appears laminar. The resulting thermal evolution and internal flow structure of the simulated thermals is remarkably similar to the DNS of initially spherical laminar thermals in LJ2019 ($R_0 = 630$). Results across the range of $A$, close to those of Lai et al. (2015), who also numerically solved the filtered (discretized) Navier–Stokes equations but using a $k-\delta$ turbulence closure (Lauder and Spaulding 1974), where $k$ is the resolved kinetic energy and $\delta$ is the energy dissipation rate. Our simulations are integrated forward in time until the thermal top (as defined in section 3b) reaches a height of $15R_0$ above the initial thermal top (i.e., top of the initial buoyancy perturbation). To investigate the internal thermal flow characteristics, particularly the time for ascent of a parcel through the thermal, each simulation includes forward trajectories for a parcel placed at the thermal bottom at the initial time. We use the built-in parcel trajectory calculations in CM1 which are done during the model integration using linear interpolation of the flow field at each model time step.

b. Analysis methodology

Thermal boundaries must first be identified and tracked in order to analyze thermal behavior including spreading rate. We use a method similar to LJ2019 and Morrison et al. (2022). At each output time (at an interval of 0.542 times the time scale $\sqrt{R_0/B_s}$), the horizontal thermal midpoint is determined by the column with maximum vertically integrated pressure perturbation. Thermal top is defined by the buoyancy field analogously to LJ2019: the provisional thermal top height $z_c$ is calculated as the highest level where the horizontally averaged $B \geq 1/10$ of the maximum horizontally averaged $B$ (maximum defined in the vertical). This is done at each output time to generate a time series of provisional $z_c$ from which we calculate the thermal top ascent rate $w_t$ using a centered difference in time. Using $z_t$ obtained directly from the $B$ field can result in noise in $w_t$ and hence in thermal volume and radius. However, unlike LJ2019 and Morrison et al. (2022), we apply $w_t$ directly to calculate the streamfunction and thermal boundaries rather than using a fitting procedure to the analytic scaling relation $w_t \approx t^{-1/2}$ (or analogously, $w_t \approx z_t^{-1}$) from similarity theory. Although the fitting method reduces noise, it is only applicable in the sticky regime after spinup, and we are interested in thermal behavior both during spinup and after. Although $w_t$ here is somewhat noisy, the spreading behavior of thermals and its sensitivity to $A$, are clear.

Once $w_t$ is determined, model output is azimuthally averaged around the horizontal midpoint using a radial–vertical grid $(r, z)$ with the same grid spacing as the original Cartesian grid. We then calculate the Stokes streamfunction using the thermal-relative flow field. This is done by integrating

$$\frac{\partial \psi}{\partial r} = 2\pi r(w_{\text{azi}} - w_t),$$

(32)

$$\frac{\partial \psi}{\partial z} = -2\pi r u_{\text{azi}},$$

(33)

where $u_{\text{azi}}$ and $w_{\text{azi}}$ are the regridded radial and vertical velocities in cylindrical coordinates, with the boundary condition $\psi(r = 0, z) = 0$. The boundary of the thermal is the $\psi = 0$ contour. Thermal radius $R$ is calculated as the widest region with $\psi \geq 0$. Spreading rate $a$ is calculated from $a = dR/dz_c$, entering centered finite differencing. Fractional entrainment rate $\epsilon$ is calculated from $d(\ln V)/dz_c$, where $V$ is defined by the volume with $\psi \geq 0$, again using centered finite differencing. Entrainment efficiency $\epsilon$ is then obtained as the product of $\epsilon$ and $R$.

Other quantities of interest are 1) vorticity, which is calculated directly from the velocity field using centered finite differencing, and 2) buoyant and dynamic components of perturbation pressure, output directly from the model as described in Morrison et al. (2022).

4. Analysis of numerical simulations

Overall structure and evolution is similar for all of the simulated thermals. Starting from rest, rapid spinup ensues owing to vorticity generation from the thermals’ buoyancy distributions. The thermals spread outward as they rise and entrain the surrounding fluid. Spinup of the thermals (after which circulation is nearly constant) occurs when a parcel initially placed at the thermal bottom rises to near proximity of the thermal top. Here, we calculate the critical height $z_c$ as the thermal top height when spinup is achieved, rather than directly from the height where $dw/dz_c = 0$ and $w_t$ is maximum because $dw_t/dz_c$ is rather noisy. Nonetheless, $z_c$ calculated from the parcel trajectories matches well with broad maxima in $w_t$ as shown later.

In accordance with the theory presented in section 2, $z_c$ ranges from about 1 to 6 as $A$, is varied from 0.5 to 2 (Table 2). After spinup, when $z_t > z_c$ in the “sticky” regime, the thermals continue to expand by entraining environmental fluid, but their overall flow structure is fairly steady. The thermals undergo a

<table>
<thead>
<tr>
<th>Feature</th>
<th>Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamics</td>
<td>Incompressible Boussinesq</td>
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<tr>
<td>Number of horizontal grid points</td>
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<tr>
<td>Number of vertical grid points</td>
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<tr>
<td>Advection</td>
<td>Fifth-order WENO</td>
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<td>Smagorinsky type</td>
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<td>Lateral boundary conditions</td>
<td>Periodic</td>
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<tr>
<td>Lower and upper boundary</td>
<td>Free slip and rigid</td>
</tr>
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</table>

Table 1. Configuration details for the CM1 simulations presented in this paper.
Table 2. Time-averaged $\alpha = dRdz_c$, $b = e/\alpha$, virtual mass parameter $C_v$, ratio of parcel to thermal-time-averaged vertical velocity $\sigma$, circulation $\Gamma$, and critical spinup height $z_c$ from the simulations with varying $A_r$. Note that $b$ is obtained from the ratio of time-averaged $e$ to time-averaged $\alpha$. Because of some noise in calculating thermal velocity directly, $\sigma$ is derived from (23) using $z_c$ obtained from the simulations as described in the text. $\alpha$, $e$, and $C_v$ are calculated as time averages over the full simulation period, whereas $\Gamma$ values are time-averaged after spinup to the end of the simulations.

<table>
<thead>
<tr>
<th>$A_r$</th>
<th>$\alpha$</th>
<th>$b$</th>
<th>$C_v$</th>
<th>$\sigma$</th>
<th>$\Gamma$</th>
<th>$z_c$</th>
</tr>
</thead>
<tbody>
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<td>0.5</td>
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<td>6.4</td>
</tr>
</tbody>
</table>

slow deceleration (relative to the faster acceleration during spinup) with $w_t$ roughly proportional to $t^{-1/2}$ (and thus also proportional to $v_c^{-1}$) in accordance with the classical similarity theory of Scolcer (1957).

Figure 4 shows vertical cross sections through the thermal center of $B$, $w$, horizontal vorticity in the $y$ direction $\eta_y$, and streamfunction $\psi$ after spinup, when thermal top height is at approximately $z_c + 2R_0$. Thermal flow features well documented by previous studies are seen in the figure. These include toroidal circulations with rotation centers near the thermal vertical midpoint, buoyancy concentrated near these rotation centers, and downward motion (in an absolute sense and relative to $w_t$) along the thermal periphery. Although buoyancy is almost entirely swept away from the thermal core (along the vertical axis at $X = 0$) for the $A_r = 0.5$ and 1 simulations, some positive buoyancy remains in the core when $A_r = 2$. There is also fluid with $B > 0$ and $\eta_y \neq 0$ below the thermal in this simulation (Figs. 4c,e,f). This occurs because not all of the initially buoyant fluid is taken into the thermal’s vortex ring (toroidal circulation) when the aspect ratio is large, a result also noted by Lai et al. (2015). This behavior can be described by the “formation number” (Gharib et al. 1998), which is related to the maximum vorticity that can be incorporated into a vortex ring before it “pinches off” from a trailing stem. Earlier work showed a formation number of 4–5 for vortex rings (Gharib et al. 1998; Wang et al. 2009), whereas Lai et al. (2015) found a somewhat lower formation number of $\sim 2$, consistent with our results. Despite the presence of a trailing stem of weakly buoyant fluid in the $A_r = 2$ simulation, buoyancy in the core is small relative to that near the rotation centers, and as detailed later, the theoretical relations between $z_c$, $\alpha$, and $A_r$ proposed in section 2 still well describe behavior of this simulation. We suspect that further increases in $A_r$ would lead to greater deviation with the theory. Indeed, Lai et al. (2015) showed little change in

$\alpha$ as $A_r$ was increased beyond 2, likely because of the inability of such thermals to incorporate all of the initially buoyant fluid. The behavior of these thermals instead resembled a starting plume, consistent with the numerical results of Bond and Johari (2010).

Differences in thermal aspect ratio with varying initial $A_r$ persist beyond spinup, although these differences are reduced compared to the initial $A_r$. The thermals with initial $A_r = 1$ become more flattened (smaller aspect ratio) during spinup. At the times shown in Fig. 4, the $A_r = 2$ simulation has an aspect ratio just slightly larger than 1, while that for $A_r = 1$ is about 0.75 and that for $A_r = 0.5$ is about 0.6. Different thermal aspect ratios among the simulations are reflected by variability in time-averaged values of $C_v$ (virtual mass parameter, see section 2); see Table 2. Here, $C_v$ is calculated at each model output time directly from the buoyancy and buoyant pressure forcing averaged over the thermal volume. Larger initial aspect ratios are associated with larger $C_v$, consistent with results from Tarshish et al. (2018). There is also an overall decrease in $C_v$ over time during spinup as the thermals flatten, particularly for the simulations with $A_r > 1$. Changes in thermal shape during spinup also lead to deviation in $b$ from the value for self-similar thermals ($b = 3$). The $A_r = 2$ thermal has the largest deviation, with $b \approx 2.06$, which is consistent with it experiencing the greatest change in aspect ratio during spinup, whereas $b$ ranges from $\sim 2.4$ to 2.7 for the other simulations.

Thermal behavior during spinup is illustrated in Fig. 5, which shows vertical cross sections of $B$, $w$, $\eta_y$, and $\psi$ in the same format as Fig. 4 except during the spinup period for the $A_r = 1$ simulation. Cross sections are shown in nondimensional time increments of 1.1 between $t = 1.6$ and 4.9. For context, the thermal top reaches $z_c$ at $t \approx 3.7$. The basic mechanism of spinup is similar for all the runs. Consistent with the discussion in section 2d, entrainment occurs as environmental fluid is swept into the thermal from below in the convergent flow. This appears as a “bite” taken from the buoyancy field from below and occurs because thermal-relative vertical velocities are strongest in the thermal core. Baroclinic vorticity generation is concentrated along the edge of the buoyancy field where there are large horizontal buoyancy gradients. Once the buoyancy field is deformed and starts to wrap around the vortex core (i.e., the center of rotation), baroclinic generation and destruction of vorticity drives a spreading of the thermal in the manner outlined by McKim et al. (2020) and Morrison et al. (2021). Flattening of the thermal during spinup is also evident in Fig. 5.

In all simulations, the thermals’ internal flow structures consist of thermal-relative ascent in the core, with strongest ascent along the vertical axis and descent along the periphery. This flow pattern strongly resembles Hill’s analytic spherical vortex. To illustrate this point further, Fig. 6 compares $w$ profiles along the thermals’ vertical axis from the simulations with $A_r$ of 0.5, 1, and 2 with $w$ profiles at the vertical axis from Hill’s vortex given by (26) and (27). This is similar to the comparison of $w$ profiles from thermal simulations with Hill’s vortex in Lai et al. (2015, Fig. 12 therein). Simulation results here are shown at the time of spinup when the thermal top is at $z_c$.

Note that this difference may be explained in part because Gharib et al. (1998) defined formation number by the maximum vorticity incorporated into the vortex ring, while Lai et al. (2015) defined it by the maximum volume of fluid incorporated.
FIG. 4. Vertical cross sections of (left) buoyancy (color contours) and vertical velocity (thin black solid lines for positive $w$ and thin black dashed lines for negative $w$; contour values are $\pm 0.1, 0.2, 0.6,$ and every $0.4$ thereafter); (right) vorticity in the $y$ plane $\eta_y$ (color contours) and streamfunction $\psi$ (contour lines). Thick black lines show thermal boundaries defined by the $\psi = 0$ isoline. Results are shown for (a),(b) $A_r = 0.5$, (c),(d) $A_r = 1$, and (e),(f) $A_r = 2$. Cross sections are shown at times when the thermal top is approximately $2R_0$ above the critical height $z_c$ for each simulation (see text).
Profiles from the simulations are normalized by the maximum \( w \) with height normalized by the thermal depth; thermal bottom and top heights are set to \(-1\) and \(1\), respectively. Correspondingly, \( a = 1 \) in (26) and (27) for the Hill’s vortex \( w \) profile. All of the simulations produce similar \( w \) profiles as Hill’s vortex, with the \( A_r = 1 \) being closest. There is also a close correspondence of the \( A_r = 2 \) simulation with Hill’s vortex, with greater deviation for \( A_r = 0.5 \). Overall, these results support the discussion in section 2e on the validity of approximating \( \sigma \) for thermals from Hill’s vortex.

Differences in spreading rate \( \alpha = dR/\text{dz} \) among the simulated thermals are seen in Fig. 7a, which shows thermal radius \( R \) as a function of \( z_t \) for the simulations with \( A_r \) of 0.5, 1, and 2. The increase of \( R \) with \( z_t \) is clearly greater as \( A_r \) is decreased, with \( \alpha \) about 3 times larger in the \( A_r = 0.5 \) simulation compared to \( A_r = 2 \). Although \( R \) is somewhat noisy, the overall spreading rates are nearly constant with \( z_t \) (seen by the dotted lines) consistent with similarity theory.

A comparison of simulated \( w_t \) as a function of \( z_t \) with solutions to the analytic \( w_t \) equation, Eq. (11), is shown in Fig. 7b. The analytic \( w_t \) are obtained using mean values of \( C_w \), \( b \), and \( \alpha \) from each simulation (Table 2). The overall behavior of \( w_t \) is similar among the simulations, with a sharp increase during spinup followed by a slower decrease after spinup. The analytic \( w_t \) are close to the simulated values for each simulation (compared the dotted and solid lines in Fig. 7). Larger values of \( \alpha \) in the \( A_r = 0.5 \) simulation correspond to faster spinup and lower height of maximum \( w_t \) (critical height \( z_c \)) compared to the \( A_r = 1 \) and especially \( A_r = 2 \) simulations with smaller \( \alpha \). Thus, \( z_c \) increases with \( A_r \) consistent with the theory in section 2d.

Thermal behavior during spinup for the simulations with \( A_r \) of 0.5, 1, and 2 is further illustrated in Fig. 8, which shows time series of thermal top height \( z_t \) and vertical velocity \( w_t \), circulation \( \Gamma \), and vertically integrated core buoyancy \( \int B_t \text{dz} \). Also shown in Fig. 8 are the height \( z_p \) and vertical velocity \( w_p \) of a parcel placed initially at the thermal bottom that rises relative to the thermal as a whole. Consistent with the discussion in previous sections, \( \Gamma \) increases during spinup owing to \( \int B_t \text{dz} > 0 \) following (13), and this is accompanied by an increase in \( w_t \). \( w_p \) increases relative to \( w_t \) as the parcel rises through the thermal core, with \( w_p \) reaching a maximum when the parcel is near the thermal’s vertical midpoint. As a result of this velocity difference, \( z_p \) increases faster than \( z_t \). Since the parcel is initiated on the thermal edge at its bottom, this marks the upward advance of entraining fluid into the thermal core (see also vertical cross sections of \( B \) in Fig. 5). This leads to a decrease in \( \int B_t \text{dz} \) and the rate of increase in \( \Gamma \) slows (i.e., \( d\Gamma/dt \) decreases). At the time when the parcel rises to near thermal top \( (z_p = z_t) \), \( \int B_t \text{dz} \) reaches steady values near \( 0 \) (though somewhat larger in the \( A_r = 2 \) simulation) and \( d\Gamma/dt \approx 0 \). This point defines the time \( \tau_t \) and height \( z_t \) of thermal spinup consistent with the discussion in section 2. The terms \( \tau_t \) and \( z_t \) are calculated here as the time and height when \( z_p \) reaches within \( 2\% \) of \( z_t \), \( \tau_t \) is denoted by the vertical black lines in Fig. 8. After spinup, \( z_p \) tracks closely to \( z_t \) and \( w_p \) remains close to \( w_t \), while both decrease slowly. Overall, \( \Gamma \) (time-averaged past spinup) increases sharply as \( A_r \) is increased, from \( \Gamma \approx 2.30 \) for \( A_r = 0.5 \) to \( \Gamma \approx 18.02 \) for \( A_r = 2 \). Note that there is a small increase in \( \Gamma \) in the \( A_r = 2 \) simulation after spinup corresponding to a small but nonnegligible \( \int B_t \text{dz} \) consistent with vertical cross sections of the \( B \) field (see Fig. 4c). This occurs because entrained fluid from below the thermal has \( B > 0 \) in this simulation; not all the initially

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**FIG. 5.** As in Fig. 4, but for the \( A_r = 1 \) simulation during spinup at the times (i) indicated.
buoyant fluid is taken up by the thermal initially when the aspect ratio is large so that some remains below the thermal’s circulation as discussed earlier. Values of $\int B_c \, dz$ reached in the $A_r = 2$ simulation after spinup appear to be nearly steady in time thereafter, and they are about an order of magnitude larger than in the other simulations after their spinup. It is expected that $\int B_c \, dz$ would eventually decrease in the $A_r = 2$ simulation as the thermal continued to rise and entrain, but investigating this would require longer simulations and thus a larger domain.

Values of $z_c$ from the simulations, estimated from $z_p$ and $z_t$ as described above, are compared to the theoretical linear $z_c - A_r$ relation (23) using $\sigma = 2$ from Hill’s analytic vortex (see section 2e) and using the average $\sigma = 1.80$ from the simulations (Table 2) in Fig. 9a. The simulated and theoretical $z_c$ values are similar, although the $A_r = 2$ simulation deviates more substantially from the theoretical relations. Reasons for this deviation are unclear but might be explained by the buoyant fluid entrained into the thermal from below in this simulation, leading to some buoyancy.

**Fig. 6.** Comparison of vertical profiles of $w$ from the simulated thermals (blue crosses at each model level) with that from Hill’s analytic spherical vortex (red lines). The thermal/vortex bottom and top heights are normalized to $-1$ and $1$, respectively, and shown by the horizontal black lines. Profiles of nondimensional $w$ are normalized such that the maximum value is $1$. Simulation results are shown for (a) $A_r = 0.5$, (b) $A_r = 1$, and (c) $A_r = 2$ near the time of thermal spinup.

**Fig. 7.** Vertical profiles of (a) thermal radius $R$ and (b) ascent rate $w$ for simulations with various $A_r$ as indicated. Solid lines show results calculated directly from the simulations. Dotted lines in (a) show fit values of constant $\alpha = dR/dz$ and in (b) show solutions to the analytic $w$ Eq. (11) using $C_d = 0$ and mean values of $C_w$, $\alpha$, $e$ from the simulations.
remaining along the thermal’s vertical axis even after spinup. The theoretical $\sigma = 2$ derived from the Eulerian mean $w$ of Hill’s vortex (see section 2c) is fairly close to $\sigma$ values obtained directly from the simulations (within 10% except for the $A_r = 2$ simulation), though somewhat larger. The simulated values range from $1.80$ to $1.95$ for $A_r < 2$ but are slightly smaller ($\approx1.63$) for $A_r = 2$ (see Table 2).

A direct comparison of the simulated and theoretical values of spreading rate $\alpha$ is shown in Fig. 9b. Theoretical values are obtained from 1) Eq. (19) using $z_c$ and $b$ derived from the simulations (Table 2), 2) Eq. (20) using $z_c$ derived from the simulations and $b = 3$ following self-similarity, 3) Eq. (25) using the average $\sigma = 1.80$ from the simulations to predict $z_c$, and 4) Eq. (31) which calculates $\alpha$ from $z_c$ predicted using $\sigma = 2$ from Hill’s analytic vortex. All of the theoretical calculations for $\alpha$ give similar results as the simulations. The simulations show a sharp decrease of $\alpha$ with increasing $A_r$ that follows an approximate $A_r^{-1}$ dependence consistent with the theoretical expressions. Using $b = 3$ instead of $b$ values obtained directly from the simulations leads to a small decrease in theoretical $\alpha$. In this case, $\alpha$ values are somewhat smaller than simulated values for $A_r \geq 1$, but closer to simulated values for $A_r < 1$. Using (31) to calculate $\alpha$ well describes the $\alpha - A_r$ relation but with $\approx10\%$ larger $\alpha$ compared to the simulations (solid line in Fig. 9b). This is consistent with the small overestimation of $\sigma = 2$ approximated from Hill’s vortex. Using the average $\sigma$ from the simulations ($\sigma = 1.80$) to predict $z_c$, and in turn $\alpha$ following (25), gives a close correspondence to the simulated $\alpha$ over the range of $A_r$ (dotted line in Fig. 9b).

5. Discussion

Overall, the simulations and theory are in reasonable agreement regarding thermal top height at spinup $z_c$ and thermal spreading rate $\alpha$ and how they vary with initial aspect ratio $A_r$. Our results indicate a nearly linear relation between $z_c$ and $A_r$ (though with greater deviation for the $A_r = 2$ simulation) and an inverse relation between $\alpha$ and $A_r$ ($\alpha \propto A_r^{-1}$). Qualitatively, this $\alpha - A_r$ relation is consistent with previous thermal studies (see Fig. 17 in Lai et al. 2015). Larger $\alpha$ is associated with greater fractional entrainment rate which leads to a lower critical height $z_c$ defined as the thermal top height when $w_t$ is maximum. $z_c$ also corresponds to the thermal top height when buoyant fluid along the thermal’s vertical axis is replaced by nonbuoyant environmental fluid entrained and advected upward through the thermal core, after which the thermal is spun up and $dV/dt \approx 0$. The time scale for this process is controlled by how long it takes for parcels initially just below the thermal bottom to ascend through the thermal, which in turn depends on $A_r$. By relating $\alpha$ to $z_c$, and $z_c$ in turn to $A_r$, we obtained the inverse relation between $\alpha$ and $A_r$ in section 2.

This explains why larger $\alpha$ is associated with smaller $A_r$, but does not by itself explain the physical mechanism. A key question, therefore, is what is the mechanism driving the increase in $\alpha$ as $A_r$ is reduced? With small $A_r$, $\int B\cdot dz$ is relatively small, and thus, $\Gamma$ increases slowly. This implies that at a given nondimensional time, $\Gamma$ will be small relative to that for a thermal with larger $A_r$. As entrained fluid rises through the thermal and sweeps out buoyant fluid along the thermal’s vertical axis, baroclinic generation and destruction of vorticity spread the vortex ring and hence thermal boundaries outward (McKim et al. 2020; also see Fig. 9 in Morrison et al. 2021). This buoyant forcing will have a relatively greater impact on the vorticity field when $\Gamma$ is small, thus leading to faster outward spread and larger $\alpha$ when $A_r$ is small. This is consistent with the impulse–circulation relation expressed by (16) after spinup when $dV/dt \approx 0$. That equation shows that for a given domain-integrated buoyant forcing $F_B$, smaller $\Gamma$ necessitates a larger increase in vortex ring radius $R_v$. Note that we cannot simply relate $\alpha$ to $A_r$ using the impulse–circulation equation.
because $\Gamma$ appears directly in this equation, and it depends on $A_t$ in a nonstraightforward way. The change in impulse over time, $dI_z/dt = F_B$, also varies with $A_t$. Moreover, $\alpha$ is defined by the change in thermal radius with thermal top height rather than over time, and thus, relating $\alpha$ directly to $dI_z/dt$ and circulation requires a transformation of variables using $d\alpha/dt = w^{-1}dI_z$. These complications motivated us to instead relate $\alpha$ to $A_t$ via $Z_c$, from which we derived the simple $\alpha \approx A_t^{-1}$ scaling as noted above. Nonetheless, relations between impulse, buoyant forcing, circulation, and thermal/vortex ring radius provide a more complete picture of the physical mechanism underpinning this simple $\alpha$-$A_t$ relation.

This work also provides a concise explanation for why initially spherical thermals ($A_t = 1$) have $\alpha \approx 0.15$ (for an unstratified, neutrally stable environment). This value of $\alpha$ is intrinsically linked to the time scale for sweeping out of the buoyancy along the thermal's vertical axis and hence thermal spinup, which itself depends on the ratio ($\sigma$) of time-averaged $w_p$ to $w_c$. The thermals' internal flow structures are similar to Hill's analytic spherical vortex, implying $\alpha \approx 2$ and in turn constraining the proportionality constant in the $\alpha \approx A_t^{-1}$ relation to $\approx 0.15$ (see section 2c).

An interesting feature is that, in a given simulation, $\alpha$ is similar before and after thermal spinup. This is evident directly from the simulations (profiles of $R$ in Fig. 7a, although they are somewhat noisy, and the vertical cross sections of thermal properties during spinup in Fig. 5), as well as indirectly by closeness of the simulated and theoretical $w$ profiles (Fig. 7b), the latter calculated assuming constant $\alpha$. Thus, $\alpha$ values are “locked in” early in the simulations, and they depend strongly on the initial conditions. Why is $\alpha$ similar during spinup and after? A possible explanation is that circulation is small early in the simulations, while at the same time, entrainment has only just begun to erode buoyancy in the thermal core. This means that buoyancy gradients and hence baroclinic generation and destruction of vorticity near the central core are weak (vorticity generation being concentrated more along the thermal boundary). However, because circulation and vorticity near the vortex core are also weak, the net result is a similar thermal spreading rate compared to later when both baroclinic generation/destruction of vorticity and circulation are stronger. Moreover, $\alpha$ is the change in $R$ with $z_c$, and small $w$ during early spinup means that a small spreading rate in time is associated with a relatively larger $\alpha$.

We also note that $\alpha$ is somewhat larger for turbulent compared to laminar thermals; LJ2019 and Morrison et al. (2022) found $\sim 20\%$ and $40\%$ larger values for turbulent thermals, respectively. At high Reynolds number, turbulent stresses lead to a spindown of circulation after thermal spinup such that $d\Gamma/dt < 0$ (Nikulin 2014; McKim et al. 2020). All else equal, $d\Gamma/dt < 0$ implies a larger spreading rate following the impulse–circulation relation (16). Nikulin (2014) developed an analytic expression for $\alpha$ as a function of $\Gamma$, $F_B$, an empirical parameter $\beta$ (encapsulating $\zeta$, $C_\alpha$, and $m$), and an empirical proportionality constant characterizing the impact of turbulent stresses. Using parameter values deduced from experimental data, they suggested a $\sim 3\%$ increase in $\alpha$ from turbulent stresses. However, this study did not consider the effects of turbulent stresses on thermals starting from rest. Reduced circulation from turbulent stresses during spinup might explain the order-of-magnitude larger impact on $\alpha$ found by LJ2019 and Morrison et al. (2022), as both studies simulated thermals that were initially motionless. The hypothesis is that turbulent stresses during spinup cause most of the differences in $\alpha$ between laminar and turbulent thermals is consistent with Fig. 3 in McKim et al. (2020), which shows that the turbulent case has $\sim 30\%$ smaller $\Gamma$ at the time of spinup relative to the laminar case.

**Fig. 9.** (a) Critical height $z_c$ and (b) thermal expansion rate $\alpha$ as functions of initial thermal aspect ratio $A_t$ from the simulations and theory. $\alpha$ and $z_c$ obtained directly from the simulations are shown by blue crosses. Green and red crosses in (b) show theoretical $\alpha$ values from (19) with $b$ obtained from the simulations or 20 with $b = 3$, respectively, with $z_c$ in both expressions obtained directly from the simulations. The theoretical $z_c$ from (23) using $\sigma = 2$ from Hill’s analytic vortex and using $\sigma = 1.8$ (average $\sigma$ from the simulations) are shown by the solid and dotted black lines, respectively. The solid and dotted black lines in (b) show theoretical $\alpha$ values from (25) using $\sigma = 2$ and 1.8, respectively.
6. Summary and conclusions

This study investigated the spreading rate \( \alpha \) and entrainment behavior of dry, buoyant thermals with varying initial aspect ratio \( A_r \). An expression was derived for the non-dimensional thermal ascent rate \( w_t \) as a function of thermal top height \( z_t \) from the thermal wind momentum budget. From this expression, we defined a critical thermal top height \( z_c \), where \( dw_t/dz_t = 0 \). The height \( z_c \) corresponds to the thermal top height when buoyancy is eroded along the thermal’s vertical axis from entrainment of nonbuoyant environmental fluid (with thermal circulation approximately constant thereafter). We then analytically solved \( dw_t/dz_t = 0 \) to derive an expression relating \( \alpha \) and \( z_c \). In turn, \( z_c \) depends on \( A_r \) and the ratio \( \sigma \) of the mean vertical velocity of a parcel rising from thermal bottom to near top along its vertical axis to \( w_t \). By approximating the thermal flow similarly to Hill’s analytic spherical vortex, it was estimated \( \sigma \approx 2 \). In this way, we derived an analytic expression for \( \alpha \) that depends inversely on \( A_r \).

Numerical simulations of thermals with \( A_r \) varying from 0.5 to 2 were analyzed and compared to the theoretical expressions. The analytic formulation for \( w_t \) well matched the thermal simulations over the range of \( A_r \). Values of \( \alpha \) calculated directly from the simulations were also close to the theoretical \( \alpha \) over the range of \( A_r \). Consistent with the theory, increasing \( A_r \) led to slower spinup owing to an increase in distance (relative to the thermal radius) for parcels to travel from thermal bottom to near top, meaning that core buoyancy was eroded more slowly by entrainment. Values of \( \sigma \) were similar among the simulations and ranged from 1.63 to 1.95, somewhat less than the theoretical \( \sigma \approx 2 \) based on the flow similarity between the thermals and Hill’s vortex. This work also provided an explanation for why initially spherical thermals \( (A_r = 1) \) have \( \alpha \approx 0.15 \), which occurs because of the similarity of thermal flow to Hill’s vortex. This gives \( \sigma \approx 2 \) and constrains the proportionality constant in the \( \alpha \propto A_r^{-1/2} \) relation to \( \sim 0.15 \). We emphasize that changes in \( z_c \) do not cause changes in \( \alpha \), but larger \( \alpha \) is associated with lower \( z_c \), and both are controlled by the erosion of buoyancy along the thermal’s vertical axis driven by entrainment of nonbuoyant fluid. This process also dictates changes in circulation that are consistent with thermal spreading rates via the thermal impulse–circulation relation.

This study has elucidated factors controlling the spreading rate of dry buoyant thermals. This work is relevant to buoyantly driven atmospheric flows, especially those with a localized pulse source of buoyancy or steady source that leads to a chain of multiple thermals. In particular, numerous studies have noted the importance of buoyant thermals for cumulus convection in the atmosphere (e.g., Blyth et al. 2005; Damiani et al. 2006; Sherwood et al. 2013; Romps and Charn 2015; Hernandez-Deckers and Sherwood 2018; Morrison et al. 2020; Peters et al. 2020). Spreading rates of dry thermals may also indirectly impact cumulus entrainment rates by influencing the size of thermals at cloud base (Mulholland et al. 2021). Although Vybhav and Ravichandran (2022) suggested similar growth rates for dry and moist (cloud) thermals, Morrison et al. (2021) found that the spreading rate of moist thermals was almost a factor of 2 smaller than dry thermals for conditions typical of cumulus convection in the lower and middle troposphere. It is unclear how results from the current study might translate to cumulus thermals, given the impact of latent heating and cooling on their buoyancy distributions. Moreover, for buoyantly driven atmospheric flows at scales of interest, dry and moist thermals are generally turbulent. Nikulin (2014) suggested that the effects of turbulent stresses can be considered as an additional term leading to a small increase in \( \alpha \). This is supported by the recent numerical modeling studies of LJ2019 and Morrison et al. (2022), although they demonstrated an order-of-magnitude larger impact on \( \alpha \) than Nikulin (2014) (~20%–40% versus a few percent). Future work should refine understanding of the entrainment behavior and spreading rates for dry and moist turbulent thermals.

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**Data availability statement.** This study used CM1 version 20.1 (cm1r20.1) released on 25 August 2020. CM1 code and detailed documentation are available at https://www2.mmm.ucar.edu/people/bryan/cm1/. Configuration and namelist files for the CM1 simulations as well as analysis code can be made available upon request to the first author.

## APPENDIX

### List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>Radius of Hill’s vortex</td>
</tr>
<tr>
<td>( A )</td>
<td>Thermal cross-sectional area</td>
</tr>
<tr>
<td>( A_r )</td>
<td>Initial thermal aspect ratio</td>
</tr>
<tr>
<td>( b )</td>
<td>Parameter defined by the ratio ( e ) to ( \alpha )</td>
</tr>
<tr>
<td>( B )</td>
<td>Buoyancy</td>
</tr>
<tr>
<td>( B_c )</td>
<td>Core buoyancy along the thermal’s vertical axis</td>
</tr>
<tr>
<td>( B_{\text{eff}} )</td>
<td>Effective buoyancy</td>
</tr>
<tr>
<td>( C_d )</td>
<td>Dynamic drag coefficient</td>
</tr>
<tr>
<td>( C_v )</td>
<td>Virtual mass parameter</td>
</tr>
<tr>
<td>( D )</td>
<td>Distance traveled by the vortex as a whole over time period ( \Delta t )</td>
</tr>
<tr>
<td>( D_0 )</td>
<td>Initial thermal vertical length</td>
</tr>
<tr>
<td>( e )</td>
<td>Entrainment efficiency</td>
</tr>
<tr>
<td>( E )</td>
<td>Momentum entrainment</td>
</tr>
<tr>
<td>( E_\alpha )</td>
<td>Euler number</td>
</tr>
<tr>
<td>( f )</td>
<td>Fractional distance from vortex center where the parcel is initiated relative to radius ( a )</td>
</tr>
<tr>
<td>( F_d )</td>
<td>Thermal-averaged pressure drag force</td>
</tr>
<tr>
<td>( F_r )</td>
<td>Froude number</td>
</tr>
<tr>
<td>( F_pB )</td>
<td>Thermal-averaged buoyant pressure drag force</td>
</tr>
<tr>
<td>( F_{pd} )</td>
<td>Thermal-averaged dynamic pressure drag force</td>
</tr>
<tr>
<td>( g )</td>
<td>Gravitational acceleration</td>
</tr>
</tbody>
</table>
I Fluid impulse
Iz Fluid impulse in the z direction
k Integration constant
k1 Integration constant
m Shape parameter defined as the ratio of V to R3
p Pressure
\( \mathbf{n} \) Unit vector normal to the thermal’s surface
u Fluid velocity vector
ue Regraded radial velocity in cylindrical coordinates
u0 Displacement rate of thermal boundary
ue Effective entrainment velocity
r Radial direction in axisymmetric coordinates
R Thermal radius
Re Reynolds number
Rc Ring vortex radius
S Region defined by circuit passing through the thermal core and returning through the ambient fluid
t Time
V Thermal volume
w Fluid vertical velocity
W Velocity of Hill’s vortex
waxi Regraded vertical velocity in cylindrical coordinates
wp Vertical velocity of a parcel along its Lagrangian path
wz Vertical velocity of thermal top
z Height
zb Height at the bottom of region S
zc Thermal top height at spinup
zt Height of thermal top
zs Height at the top of region S
\( \alpha \) Rate of increase in thermal radius with height as the thermal rises, equivalent to \( dR/dz \)
\( \gamma \) Thermal shape parameter defined as the ratio \( AR/V \)
\( \Gamma \) Thermal circulation
\( \Delta L_m \) Grid spacing of the numerical model
\( \Delta t \) Time for parcel to travel from near vortex bottom to near its top
\( \varepsilon \) Fractional entrainment rate
\( \eta \) Horizontal vorticity in the y direction
\( \omega_\phi \) Azimuthal vorticity
\( \Omega \) Region of space occupied by thermal
\( \psi \) Streamfunction
\( \theta \) Potential temperature
\( \rho \) Fluid density
\( \rho_0 \) Constant background fluid density
\( \sigma \) Ratio of time-averaged vertical velocities of the parcel and thermal top
\( \tau_c \) Time scale for thermal top to reach \( z_c \)
\( \zeta \) Ratio of ring vortex radius to thermal radius

REFERENCES


